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CATEGORICAL W^*-TENSOR PRODUCT

BY
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Abstract. If A and B are von Neumann algebras and $A \overline{\otimes} B$ denotes their categorical C^*-tensor product with the universal property, then the von Neumann tensor product $A \nabla B$ of A and B is defined as

$$A \nabla B = (A \overline{\otimes} B)^{**}/J,$$

where $J \subset (A \overline{\otimes} B)^{**}$ is an appropriate ideal. It has the universal property.

1. **Introduction.** Consider the category $\boldsymbol{C^*}$ of C^*-algebras and the category $\boldsymbol{W^*}$ of von Neumann algebras. There is a functor $E: \boldsymbol{C^*} \to \boldsymbol{W^*}$ that assigns to a C^*-algebra A its universal enveloping algebra $EA = A^{**}$, or double dual. A C^*-algebra A is a von Neumann (or W^*) algebra if and only if A is the dual $A = A_*^*$ of some unique Banach space A_* called the predual. The preduals form a category $\mathfrak{B}$ of Banach spaces that is anti-isomorphic to $\boldsymbol{W^*}$ under the contravariant functor $P: \boldsymbol{W^*} \to \mathfrak{B}$ given by $PA = A_*$.

An example at the end (see Example 6.3) shows that the usual tensor product on $\boldsymbol{W^*}$ does not have the universal property. One of the objectives here is to construct a new tensor product "$\_\nabla\_$" on $\boldsymbol{W^*}$ that has the universal property. The construction is carried out without representing the algebras as operators on some (nonunique) Hilbert spaces, but rather more conceptually by first constructing a product "$\_\triangle\_$" on $\mathfrak{B}$ and then transferring it as "$\_\nabla\_$" to the dual category $\boldsymbol{W^*}$ by use of the functors P and E. A tensor product $A \overline{\otimes} B$ of C^*-algebras is said to have the universal property provided whenever $\alpha: A \to D$ and $\beta: B \to D$ are C^*-maps with pointwise commuting images in D, then there is a unique C^*-map $(\alpha, \beta): A \overline{\otimes} B \to D$ giving the usual commutative diagram. It has long been known that the subcategory $\boldsymbol{C^*1} \subset \boldsymbol{C^*}$ of C^*-algebras with identity and identity preserving morphisms has a tensor product. In this setting it will be shown that

$$(\boldsymbol{C^*1}, \overline{\otimes}) \xrightarrow{E} (\boldsymbol{W^*}, \nabla) \xrightarrow{P} (\mathfrak{B}, \triangle)$$

are functors of multiplicative categories, where the morphisms in $\boldsymbol{C^*1}$ need not preserve identities.

Consider two von Neumann algebras $A = A'' \subset \mathscr{L}(H)$ and $B = B'' \subset \mathscr{L}(K)$, where

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A'' is the double commutant of A and $\mathscr{L}(H)$ denotes all bounded operators on a Hilbert space H. The usual definition of the von Neumann tensor product is $(A \bigcirc B)'' \subset \mathscr{L}(H \overline{\otimes} K)$, where "$\_\bigcirc\_$" will always denote the algebraic tensor product. Only as long as A and B are represented on some Hilbert spaces such that $A = A''$ and $B = B''$, will this definition be independent of the Hilbert spaces H and K. Two different proofs of this independence have been given by Misonou [9, p. 190, Theorem 1] and Takeda [15, p. 218, Theorem 3]. The exact same tensor product has been defined more abstractly by use of preduals by Sakai [10, p. 3.17, Theorem 2.3]. Authors treating duality of locally compact groups and Hopf W^*-algebras have consistently used the above noncategorical tensor product. Perhaps duality theory would assume a more elegant form if the right tensor product was used.

2. **Product on the preduals.**

2.1. For any C^*-algebras A, B with or without identities, consider all C^*-homomorphisms $\alpha: A \to D$ and $\beta: B \to D$ with pointwise commuting images into some C^*-algebra D. If $m: D \bigcirc D \to D$ denotes the map defined by the multiplication of D, let ρ denote the norm

$$\rho(t) = \sup \{\|m(\alpha \bigcirc \beta)(t)\| : \alpha, \beta, D\}, \qquad t \in A \bigcirc B,$$

on the algebraic tensor product $A \bigcirc B$, where the supremum is taken over all such α, β, and D. That ρ actually is a cross norm, i.e. that $\rho(a \circ b) = \|a\| \, \|b\|$, is shown in [6, p. 11, 3.1]. Let $A \overline{\bigcirc} B$ be the ρ-closure of $A \bigcirc B$.

Only in case both A and B have identities, define $A \overline{\otimes} B$ as $A \overline{\otimes} B = A \overline{\bigcirc} B$. In case $1 \notin A$ or $1 \notin B$ or both, the functor "$\_\overline{\otimes}\_$" later will be given a different meaning and definitely will not denote the above completion. If A and B have identities, then $A \overline{\otimes} B$ is the categorical tensor product for C^*-algebras, i.e. the above maps α and β factor through a unique homomorphism $\gamma: A \overline{\otimes} B \to D$. Define (α, β) to be $(\alpha, \beta) = \gamma$. Note that D may or may not have an identity and in case it does, the maps α, β may or may not preserve identities.

2.2. Any von Neumann algebra A is the dual $A = A_*^*$ of a unique Banach space A_* called its *predual*. The dual of A is A^* and $\sigma(A, A_*)$ denotes the weakest or smallest topology on A making all the functionals of A_* continuous. In case A is understood, simply write $\sigma = \sigma(A, A_*)$. A map between von Neumann algebras (not necessarily a homomorphism) is *normal* provided it preserves least upper bounds of arbitrary sets of positive elements. Then the σ-continuous linear functionals on A are precisely A_*; moreover, A_* can be intrinsically characterized as exactly the normal functionals [10, p. 121, Theorem 5.1].

2.3. The norm ρ induces a positive extended real valued function ρ' on $A^* \bigcirc B^*$ defined at $f \in A^* \bigcirc B^*$ as the supremum of the function f over the ρ-unit sphere of $A \bigcirc B$ by

$$\rho'(f) = \sup \{|\langle f, t\rangle| : t \in A \bigcirc B, \rho(t) = 1\},$$

where $\langle\_,\_\rangle$ will subsequently denote the dual pairing between a space and a space of linear functionals on it, i.e. a bilinear map $\langle\_,\_\rangle: A^* \bigcirc B^* \times A \bigcirc B \to \mathbf{C}$. There is a least cross norm λ on $A \bigcirc B$ whose dual λ' is also a cross norm (see [11]). Since, firstly, ρ is a cross norm and since, secondly, $\rho \geqq \lambda$, it follows also that ρ' is a cross norm. The completion of the algebraic tensor product $A_* \bigcirc B_*$ in the norm ρ' will be denoted as $A_* \overline{\bigcirc} B_*$. Again, if both A and B have identities, write $A_* \overline{\otimes} B_* = A_* \overline{\bigcirc} B_*$. Since by definition, the norm ρ' when extended to $A_* \overline{\otimes} B_*$ is the operator norm of $A \bigcirc B \to C$, it follows that $A_* \overline{\otimes} B_* \subset (A \overline{\otimes} B)^*$. Subsequently of main interest will be a space between the latter two.

2.4. The normed space $(A \overline{\otimes} B)^*$ will be identified with a suitably normed space of linear maps $T: A \to B^*$.

For $T \in (A \overline{\otimes} B)^*$ and $a \in A$, let $T(a \otimes \_) \in B^*$ be defined by

$$T(a \otimes \_): B \to C, \qquad b \to T(a \otimes b), \qquad b \in B.$$

Thus T induces linear maps

$$\begin{array}{ll} A \to B^*, & B \to A^*, \\ a \to T(a \otimes \_), & b \to T(\_ \otimes b), \end{array}$$

where $\langle T(a \otimes \_), b\rangle = T(a \otimes b) = \langle a, T(\_ \otimes b)\rangle$.

Conversely, any arbitrary linear map $T: A \to B^*$ can be regarded as a linear functional on $A \bigcirc B$ by means of the definition $T(\sum (a \bigcirc b)) = \sum \langle Ta, b\rangle$. Define an extended real valued norm $|||T|||$ on these linear maps T by

$$|||T||| = \sup\left\{ \left|\sum \langle Ta, b\rangle\right| : \sum (a \bigcirc b) \in A \bigcirc B, \rho\left(\sum (a \bigcirc b)\right) = 1\right\}.$$

Note that the sup-norm of the functional T on the normed space $(A \bigcirc B, \rho)$ is precisely $|||T|||$. Thus the condition that T be extendable to a continuous functional $T: A \overline{\otimes} B \to C$ is that $|||T|||$ be finite in which case $|||T|||$ is simply the norm of the extended functional. Hence $(A \overline{\otimes} B)^*$ can be identified as the space of those linear maps $T: A \to B^*$ that are finite in the norm $|||T|||$. In addition, the norm $|||T|||$ is the norm of T as a linear functional on $A \overline{\otimes} B$ (see [11, p. 45, Theorem 3.1]). Alternatively one could consider also $T: B \to A^*$.

The previous characterization of $(A \overline{\otimes} B)^*$ allows us to define a product $A_* \triangle B_*$ of the preduals of A and B as a space of linear maps $T: A \to B^*$ whose range is suitably restricted.

Definition 2.5. For von Neumann algebras A and B, define $A_* \triangle B_*$ as the closed subspace in $A_* \triangle B_* \subset (A \overline{\otimes} B)^*$ consisting of all those $T \in (A \overline{\otimes} B)^*$ such that $T: A \to B^*$ with $TA \subseteq B_*$ and also such that when T is viewed as a map $T: B \to A^*$, then $TB \subseteq A_*$.

Alternatively, $A_* \triangle B_*$ is the set of all $T \in (A \overline{\otimes} B)^*$ such that $T(a \otimes \_) \in B_*$ and $T(\_ \otimes b) \in A_*$ for all $a \in A$, $b \in B$.

3. **Universal enveloping algebras.** In addition to providing a suitable framework for constructing the von Neumann tensor product, it will be shown, among other

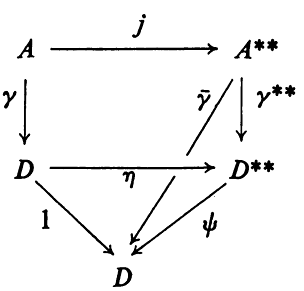
things, that the universal enveloping algebra A^{**} of an infinite dimensional W^*-algebra A contains at least two distinct isomorphic copies of A.

3.1. Consider a Banach algebra A, elements $a, b, x \in A$; $p \in A^*$; and $F, G \in A^{**}$. Then

$$\begin{aligned} apb \in A^* &: (apb)(x) = p(bxa); \\ Gp \in A^* &: (Gp)(a) = G(pa); \\ pG \in A^* &: (pG)(a) = G(ap); \\ FG \in A^{**} &: (FG)(p) = F(Gp). \end{aligned}$$

With this multiplication, A^{**} becomes a Banach algebra containing A as a subalgebra $A \subset A^{**}$. For any homomorphism $\alpha: A \to D$ into a Banach algebra D, the natural map $\alpha^{**}: A^{**} \to D^{**}$ is also a homomorphism; moreover, the restriction (and corestriction) $\alpha^{**}|A$ of α^{**} to A (and D) is $\alpha^{**}|A = \alpha$.

3.2. If, in addition, A is a C^*-algebra, then A^{**} is its *universal* (von Neumann) enveloping algebra. The algebra A^{**} together with the natural inclusion $j: A \to A^{**}$ is uniquely characterized by the universal property that any C^*-map $\gamma: A \to D$ of A into any von Neumann algebra D factors uniquely through $\bar{\gamma}: A^{**} \to D$. If $\eta: D \to D^{**}$ is the natural inclusion, then by the universal property of D^{**} there exists a unique normal map $\psi: D^{**} \to D$ with $\psi\eta = 1$, the identity on D. Then there is a commutative diagram



The map γ^{**} is well known to be normal while η in general is not normal. Although the maps γ^{**} and $\eta\bar{\gamma}$ agree on $j(A)$, in general they need not be equal.

Some already known facts (see [10]) are organized in the next lemma in a form in which they will be used later.

LEMMA 3.3. *Suppose A is a C^*-algebra with or without identity and $W \subset A^*$ a norm closed subspace invariant under translations from A. If $W^\perp$ is defined to be the polar*

$$W^\perp = \{F \in A^{**} \mid FW = \{0\}\},$$

then

(i) $A^{**} = A^{**}(1-e) \oplus W^\perp$ *where both summands are ideals and von Neumann subalgebras; e is a central idempotent in A^{**} and an identity for $W^\perp$;*

(ii) $A^* = W \oplus eA^*$; *furthermore* $W = (1-e)A^*$.

There are natural isomorphisms induced by the direct sum in (ii) *and the natural action of* A^{**} *on* A^*:

(iii) $W^* \cong A^{**}(1-e)$;

(iv) $(eA^*)^* \cong W^\perp$.

If $D = D_*^*$ *is any von Neumann algebra and* $A = D$, $W = D_*$ *above, then*

(v) $D \cong D^{**}(1-e)$.

Proof. (i) It follows from [10, p. 1.74, Corollary 15.1] that $W^\perp \triangleleft A^{**}$. From now on the notation "$\triangleleft$" always will indicate a (not necessarily closed) two sided ideal. Furthermore, $W^* \cong A^{**}/W^\perp$. Consequently, $A^{**}/W^\perp$ is a von Neumann algebra with predual W. It follows from [10, p. 1.74, Remark 2] and [10, p. 2.3, Theorem 1.3] that $W^\perp$ is a $\sigma(A^{**}, A^*)$-closed ideal and is itself a von Neumann algebra. Every von Neumann algebra $W^\perp$ contains an identity $e \in W^\perp \subset A^{**}$. Since $e \in W^\perp \triangleleft A^{**}$, for any $F \in A^{**}$, $eF = e^2F = eFe \in W^\perp$. Hence $eF = eFe = Fe$, and e is a central idempotent of A^{**}. Thus $A^{**} = A^{**}(1-e) \oplus W^\perp$.

(ii) For any closed subspace $W \subset A^*$, the kernel of $W^\perp$ (also called the double annihilator of W) is W, i.e.

$$\{f \in A^* \mid \langle W^\perp, f\rangle = \{0\}\} = W.$$

First clearly $W \subseteq$ kernel $W^\perp$. On the other hand, consider $f \in$ kernel $W^\perp$. If $f \notin W$, then by the Hahn-Banach theorem there exists $G \in A^{**}$ with $GW = \{0\}$ but $G(f) \neq 0$. Thus $G \in W^\perp$, $f \in$ kernel $W^\perp$, but $G(f) \neq 0$, a contradiction. Since $W^\perp = A^{**}e$, $(1-e)A^* \subseteq$ kernel $W^\perp = W$. But

$$\{0\} = \langle W^\perp, eA^* \cap W\rangle = \langle A^{**}e, eA^* \cap W\rangle = \langle A^{**}, eA^* \cap W\rangle.$$

Since $eA^* \cap W$ is annihilated by all of A^{**}, it follows that $eA^* \cap W = \{0\}$ and that $A^* = W \oplus eA^*$. Thus $W = (1-e)A^*$. It should be noted that since e is a central idempotent, $eA^* = A^*e$.

(iii) and (iv). Since $A^{**} = W^\perp \oplus A^{**}(1-e)$, clearly $W^* \cong A^{**}/W^\perp \cong A^{**}(1-e)$ and $(eA^*)^* \cong A^{**}/(eA^*)^\perp$. Since $(1-e)e = 0$, $A^{**}(1-e) \subseteq (eA^*)^\perp$. Since kernel $W^\perp = W$, a nonzero element of $W^\perp$ cannot vanish on all of eA^* without vanishing on all of $A^* = W \oplus eA^*$ and thus being identically zero. Thus $(eA^*)^\perp = A^{**}(1-e)$ and $(eA^*)^* \cong W^\perp$.

(v) Now specialize the above A to a von Neumann algebra $A = D = D_*^*$ and $W = D_*$ to be the predual. Then $D^{**} = D^{**}(1-e) \oplus D_*^\perp$ is an ideal direct sum of von Neumann algebras, where $D_*^* \cong D^{**}(1-e) \cong D$.

3.4. Remark. A positive linear functional on a von Neumann algebra D is called *singular* provided it annihilates arbitrarily small projections in the partial order on D (see [10, p. 1.75, Remark 2]). The space of singular linear functionals is precisely eD^*.

3.5. Associated with each pair W, A are two intrinsic maps—the natural inclusion $\eta: A \to A^{**}$ and the projection $\pi: A^{**} \to A^{**}(1-e)$ onto the first factor.

Later, when necessary, the functorial nature of these maps will be indicated by η_A and π_A.

The author knows of no place in the literature where either the next proposition or corollary is proved.

PROPOSITION 3.6. *Consider any C^*-algebra (with or without identity) and any norm closed translation invariant subspace $W \subset A^*$. If W separates the points of A, i.e. W, A are nonsingularly paired, then $\eta A \cap A^{**}e = \{0\}$ and the map $\pi\eta \colon A \to A^{**}(1-e)$ is monic.*

Proof. By definition of $\eta \colon A \to A^{**}$, for $\eta a \in A$, we have $\langle \eta a, W\rangle = \langle W, a\rangle$. Thus the spaces ηA and W are nonsingularly paired. It follows that

$$\langle A^{**}e, W\rangle = \{0\} \Rightarrow \eta A \cap A^{**}e = \{0\}.$$

Finally, since η is monic, since kernel $\pi = A^{**}e$, and $\eta A \cap A^{**}e = \{0\}$, it follows that $\pi\eta \colon A \to A^{**}(1-e)$ is monic.

COROLLARY 3.7. *If in the above proposition $A = D = D_*^*$ is a von Neumann algebra with $W = D_*$, the predual, then $\pi\eta$ is an isomorphism and there exists a normal isomorphism of von Neumann algebras θ with*

$$D \xrightarrow{\eta} D^{**} \xrightarrow{\pi} D^{**}(1-e) \xrightarrow{\theta} D$$

and such that $\theta\pi\eta = 1$, the identity on D. Furthermore $\theta\pi = \psi$, where $\psi \colon D^{**} \to D$ is the unique normal map with $\psi\eta = 1$ on D.

Proof. By 3.6, $\pi\eta$ is monic. It is asserted that $\pi\eta(D) = D^{**}(1-e)$. Suppose not. Let $F \in D^{**}(1-e) \backslash \pi\eta(D)$. Since $F \in D^{**}(1-e) \cong D_*^*$ and $D_*^* \cong D$, there necessarily exists an element $d \in D$ with $\langle F, f\rangle = \langle f, d\rangle$ for all $f \in D_*$. But $\langle f, d\rangle = \langle \eta d, f\rangle$. Consequently, $\langle F - \eta d, D_*\rangle = \{0\}$ and $F - \eta d \in D_*^{\perp} = D^{**}e$. Thus

$$F - \eta d = (F - \eta d)e = -(\eta d)e \in \eta D \cap D^{**}e = \{0\}$$

by 3.6, a contradiction.

Since each element $z \in D^{**}(1-e)$ is uniquely of the form $z = (\eta d)(1-e)$ with $d \in D$, define a C^*-isomorphism $\theta \colon D^{**}(1-e) \to D$ by $\theta z = d$. However, any isomorphism of von Neumann algebras is automatically normal. Thus there are maps

$$D \xrightarrow{\pi\eta} D^{**}(1-e) \xrightarrow{\theta} D$$

with $\pi\eta$ monic, with θ an isomorphism and with $\theta\pi\eta = 1$. Since both ψ and $\theta\pi$ are normal maps with $\theta\pi\eta = \psi\eta = 1$ on D, by the uniqueness of the map $D^{**} \to D$ for the universal enveloping algebra of D, it follows that $\theta\pi = \psi$.

REMARK 3.8. If $D = D_*^*$ is an infinite dimensional W^*-algebra, then $D_* \neq D^*$, and hence $D_*^{\perp} = D^{**}e \neq \{0\}$. Then $D \cong \eta D \subset D^{**}$ and also $D \cong D^{**}(1-e) \subset D^{**}$ are two distinct isomorphic copies of D inside D^{**}. They are distinct because D^{**}

is the σ-closure of ηD and $D^{**}(1-e)$ is a von Neumann subalgebra of D^{**}, i.e. since $D^{**}(1-e)$ is σ-closed, $\eta D \subseteq D^{**}(1-e)$ would imply that the σ-closure of ηD is also in $D^{**}(1-e)$.

4. **The von Neumann tensor product.** The tensor product defined below is different from the usual one in [10]; the one in [10] is also equivalent to that considered in [9] and [14].

4.1. Consider von Neumann algebras A and B. Since A_* and B_* are translation invariant, $A_* \triangle B_*$ is invariant under translations from $A \bigcirc B$ and hence also $A \,\overline{\otimes}\, B$. Clearly, $A_* \,\overline{\otimes}\, B_* \subseteq A_* \triangle B_*$.

DEFINITION 4.2. For any von Neumann algebras A, B define their tensor product $A \bigtriangledown B$ as the von Neumann algebra

$$A \bigtriangledown B \equiv (A \,\overline{\otimes}\, B)^{**}/(A_* \triangle B_*)^{\perp}.$$

(Since $(A_* \triangle B_*)^* \cong A \bigtriangledown B$, we have $(A \bigtriangledown B)_* = A_* \triangle B_*$.)

In the next lemma as well as throughout this section it is assumed that homomorphisms preserve identities.

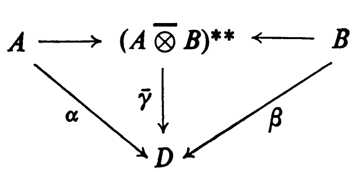
LEMMA 4.3. *Suppose $\alpha: A \to D$ and $\beta: B \to D$ are normal identity preserving maps of three von Neumann algebras A, B, D with pointwise commuting images. Consider the resulting map $\gamma: A \,\overline{\otimes}\, B \to D$ on the C^*-tensor product and the induced maps $\gamma^*: D^* \to (A \,\overline{\otimes}\, B)^*$, $\alpha_*: D_* \to A_*$, $\beta_*: D_* \to B_*$. Then $\gamma^* D_* \subseteq A_* \triangle B_*$.*

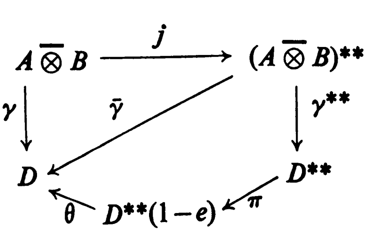
Proof. Suppose $p \in D_*$. Then $\gamma^* p: A \,\overline{\otimes}\, B \to \mathbf{C}$. It suffices to show that for any fixed $a \in A$, the linear functional $\gamma^* p(a \otimes \_): B \to \mathbf{C}$ is in B_*. The translate $p\alpha(a)$ of p is also in D_*. Thus $\gamma^* p(a \otimes \_) = [p\alpha(a)]\beta$, where

$$B \xrightarrow{\ \beta\ } D \xrightarrow{\ p\alpha(a)\ } \mathbf{C}$$

is a composition of the two normal maps β and $p\alpha(a)$. Hence their composite $[p\alpha(a)]\beta$ is also a normal map and $\gamma^* p(a \otimes \_) \in B_*$. The argument for showing that $\gamma^* p(\_ \otimes b) \in A_*$ is entirely parallel and is omitted.

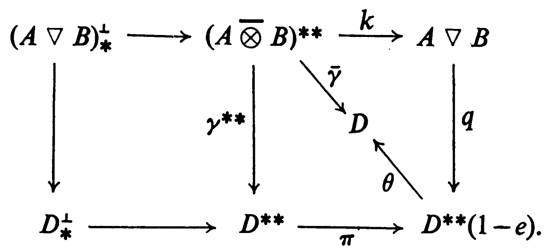
LEMMA 4.4. *Assume the hypotheses of Lemma 4.3 and denote by $\bar{\gamma}$ the unique extension of $\gamma = (\alpha, \beta)$ to the enveloping algebra $(A \,\overline{\otimes}\, B)^{**}$. Suppose $j: A \,\overline{\otimes}\, B \to (A \,\overline{\otimes}\, B)^{**}$ and $\eta: D \to D^{**}$ are the natural inclusions while π and θ are the maps of 3.7. Then $\gamma^{**} j = \eta\gamma$, $\bar{\gamma} = \theta\pi\gamma^{**}$, and there are commutative diagrams*





Proof. The first diagram is obtained by combining the commutative diagrams of $A \overline{\otimes} B$ and $(A \overline{\otimes} B)^{**}$. Since by 3.2, $\eta\gamma = \gamma^{**}j$, we have $\theta\pi\eta\gamma = \theta\pi\gamma^{**}j$. But use of 3.7, i.e. the fact that $\theta\pi\eta = 1$, shows that $\gamma = \theta\pi\gamma^{**}j$. Thus the second diagram also commutes.

PROPOSITION 4.5. *Under the hypotheses of Lemma* 4.3, *there is a normal map q and a commutative diagram*



Proof. Quite generally, even for Banach spaces

$$\gamma^*(D_*) \subseteq A_* \triangle B_* \Rightarrow \gamma^{**}((A_* \triangle B_*)^\perp) \subseteq D_*^\perp.$$

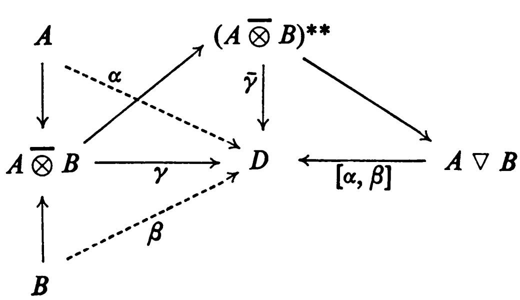
But by definition of $A \triangledown B$, $(A \triangledown B)_* = A_* \triangle B_*$. Since both $(A \triangledown B)_*^\perp \subset (A \overline{\otimes} B)^{**}$ and $D_*^\perp \subset D^{**}$ are von Neumann subalgebras as well as ideals, the quotients are von Neumann algebras and the induced map

$$q\colon (A \overline{\otimes} B)^{**}/(A \triangledown B)_*^\perp \to D^{**}/D_*^\perp$$

is also normal. But the first algebra is $A \triangledown B$ by definition, while $D^{**}/D_*^\perp \cong D^{**}(1-e)$ by 3.3(v). If k is the normal map $k\colon (A \overline{\otimes} B)^{**} \to A \triangledown B$, then it follows from the definition of q that $\pi\gamma^{**} = qk$. Thus the above diagram commutes.

The next corollary follows immediately upon combining the diagram for the C^*-tensor product with the previous proposition.

COROLLARY 4.6. *There is a commutative diagram of normal maps with* $[\alpha, \beta] = \theta q$:



At the expense of omitting the proof of the uniqueness of $[\alpha, \beta]\colon A \triangledown B \to D$, the reader can omit the topological considerations (ii) and (iii) in the next lemma without interrupting the continuity of the rest of the material. Later 4.7(i) is improved in 5.8.

LEMMA 4.7. *Consider von Neumann algebras A, B; their algebraic and von Neumann tensor products $A \bigcirc B$, $A \bigtriangledown B$.*

(i) *There is a natural embedding $A \bigcirc B \subset A \bigtriangledown B$.*

(ii) *Furthermore, $A \bigcirc B$ is $\sigma(A \bigtriangledown B, (A \bigtriangledown B)_*)$-dense in $A \bigtriangledown B$.*

(iii) *Suppose α and β factor through a normal map $g: A \bigtriangledown B \to D$. Then $g=[\alpha, \beta]$.*

Proof. (i) First it will be shown that for any $0 \neq t \in A \bigcirc B$, there exists $p \bigcirc q \in A_* \bigcirc B_*$, where p and q are normal, such that $\langle p \bigcirc q, t\rangle \neq 0$. Assume $A \subset \mathscr{L}(H)$ and $B \subset \mathscr{L}(K)$ are von Neumann algebras on Hilbert spaces H and K. Since $0 \neq t \in A \bigcirc B \subset \mathscr{L}(H \overline{\otimes} K)$, it follows that $t(h \otimes k) \neq 0$ for some $h \in H$, $k \in K$. Thus $(t(h \otimes k), x \otimes y) \neq 0$ for some $x \in H$, $y \in K$. Define $p \in A_*$ by $pa=(ah, x)$, and $q \in B_*$ similarly as $qb=(bk, y)$. Then

$$\langle p \bigcirc q, t\rangle = (t(h \otimes k), x \otimes y) \neq 0.$$

Consequently $p \bigcirc q \in A_* \bigcirc B_* \subset A_* \bigtriangleup B_*$ and $A \bigcirc B \cap (A_* \bigtriangleup B_*)^{\perp} = \{0\}$. Thus there is an embedding

$$A \bigcirc B \cong \frac{A \bigcirc B + (A_* \bigtriangleup B_*)^{\perp}}{(A_* \bigtriangleup B_*)^{\perp}} \subset A \bigtriangledown B.$$

(ii) Any Banach space E is $\sigma(E^{**}, E^*)$-dense in E^{**}; hence $A \overline{\otimes} B$ is $\sigma((A \overline{\otimes} B)^{**}, (A \overline{\otimes} B)^*)$-dense in $(A \overline{\otimes} B)^{**}$. Since $A \overline{\otimes} B$ was defined as the norm closure of $A \bigcirc B$, any functional on $A \overline{\otimes} B$ is completely determined by its value on $A \bigcirc B$. From this it follows that $A \bigcirc B$ is a $\sigma((A \overline{\otimes} B)^{**}, (A \overline{\otimes} B)^*)$-dense subset of $(A \overline{\otimes} B)^{**}$. Now $A_* \bigtriangleup B_*$ may be regarded as linear functionals on $(A \overline{\otimes} B)^{**}/(A_* \bigtriangleup B_*)^{\perp} = A \bigtriangledown B$. Then $\sigma(A \bigtriangledown B, A_* \bigtriangleup B_*)$ is the quotient topology on $A \bigtriangledown B$. Thus $A \bigcirc B \subset A \bigtriangledown B$ remains dense after passage to quotients.

(iii) A C^*-homomorphism $g: A \bigtriangledown B \to D$ is normal if and only if it is σ-continuous (see [10, p. 1.21, Theorem 5.1]). By the universal property of the C^*-tensor product, if $\bar{g}=g|A \overline{\otimes} B$ is the restriction, then $\gamma=\bar{g}=[\alpha, \beta]|A \overline{\otimes} B$. Thus since both $[\alpha, \beta]$ and g agree on a σ-dense set, $[\alpha, \beta]=g$.

The previous results are summarized in the next theorem and all the topological parts are isolated out in a corollary.

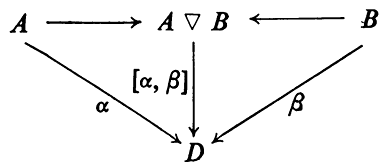
4.8. THEOREM I. *Consider von Neumann algebras $A=A_*^*$, $B=B_*^*$, their von Neumann tensor product $A \bigtriangledown B$, and the product $A_* \bigtriangleup B_*$ of their preduals. Then*

(i) *there is a natural embedding of the algebraic tensor product in $A \bigcirc B \subset A \bigtriangledown B$;*

(ii) *the predual of $A \bigtriangledown B$ is $A_* \bigtriangleup B_*$.*

Suppose $\alpha: A \to D$ and $\beta: B \to D$ are any normal identity preserving C^*-homomorphisms with pointwise commuting images into any von Neumann algebra D.

(iii) *Then there exists a unique normal C^*-homomorphism* $[\alpha, \beta]$ *giving a commutative diagram*

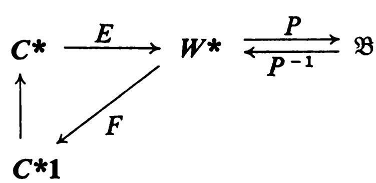


4.9. Corollary to Theorem I. *Under the above hypotheses,* $A \bigcirc B \subset A \bigtriangledown B$ *is* $\sigma(A \bigtriangledown B, (A \bigtriangledown B)_*)$*-dense.*

5. **Functors on C^*-algebras.** The von Neumann tensor product has been developed with an eye towards constructing a duality theory for locally compact (nonabelian) groups that would supplement known results (e.g. [5]) in terms of multiplicative functors, cogebras, and bigebras (e.g. [8] and [13]). Since the indispensable algebra $L^1(G)$ does not contain an identity unless the group G is compact, it becomes absolutely necessary to construct a tensor product for C^*-algebras irrespective of whether they do or do not have identity elements. Then it will be shown that the products $\overline{\otimes}$, $\bigtriangledown$, and $\bigtriangleup$ commute with certain natural functors.

Notation 5.1. Consider the category $\boldsymbol{C^*}$ of C^*-algebras and the full subcategory $\boldsymbol{C^*1} \subset \boldsymbol{C^*}$ of those with identity. The von Neumann algebras, or W^*-algebras, and normal C^*-homomorphisms form a subcategory $\boldsymbol{W^*} \subset \boldsymbol{C^*1}$.

Next, four functors E, F, P, and P^{-1} arc defined. If E: $\boldsymbol{C^*} \to \boldsymbol{W^*}$ is defined by $EA = A^{**}$, then let F: $\boldsymbol{W^*} \to \boldsymbol{C^*}$ be the forgetful functor. For $A \in \boldsymbol{W^*}$, set $PA = A_*$. Let $\mathfrak{B}$ be the category of involutive Banach spaces U with $U^* \in \boldsymbol{W^*}$; morphisms are maps f: $U \to V$ of involutive Banach spaces such that f^*: $V^* \to U^*$ is in $\boldsymbol{W^*}$. Thus P is a contravariant functor P: $\boldsymbol{W^*} \to \mathfrak{B}$ making $\boldsymbol{W^*}$ and $\mathfrak{B}$ dual categories. For $U \in \mathfrak{B}$, let $P^{-1}U = U^* \in \boldsymbol{W^*}$ be the inverse of P. It should be noted that $\boldsymbol{C^*1}$ and $\boldsymbol{W^*}$ contain the subcategories of algebras with identity and identity preserving maps; there is also a corresponding such subcategory of $\mathfrak{B}$. The above categories and functors are summarized in the following diagram:



For the sake of simplicity, notation that by now is standard (see [8] and [13]) will be used without further unnecessary definitions and explanations.

5.2. In order to show that E is a left adjoint of F, define a set functor α: $\boldsymbol{W^*}(E\_, \_) \cong \boldsymbol{C^*}(\_, F\_)$ by

$$\alpha_{AM} : \boldsymbol{W^*}(EA, M) \to \boldsymbol{C^*}(A, FM),$$
$$n : A^{**} \to M, \qquad \alpha_{AM}(n) = n|A,$$

where $n|A$ denotes the restriction. For $c \in C^*(A, FM)$, $\alpha_{AM}^{-1}(c) = \bar{c}$ where c factors through $\bar{c}: A^{**} \to M$ by the universal property of A^{**}.

The front adjunction η is a natural transformation of the identity functor $1(C^*)$ of C^* to $\eta: 1(C^*) \to FE$ defined by

$$\eta_A \equiv \alpha_{A,EA}(1_{EA}): A \to A^{**}.$$

In this case it is simply the natural inclusion.

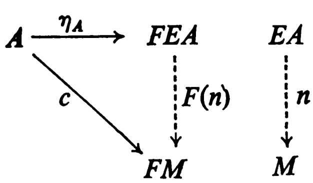
The back adjunction is a functor $\psi: EF \to 1(W^*)$ defined as

$$\psi_M = \alpha_{FM,M}^{-1}(1_{FM}): EFM \to M.$$

For a von Neumann algebra $M = M_*^*$, in the notation of the previous section, $M_*^{\perp} = M^{**}e$ and ψ is simply the composition $\psi = \theta\pi$ where $M^{**} \to M^{**}(1-e) \to M$ with $M^{**}(1-e) \cong M$.

The main properties of η and ψ will next be described.

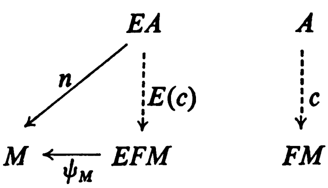
5.3. If we only wanted to prove that E is a left adjoint of F, it would simply suffice to define the natural transformation η as above and then show that for any (notation: $\forall$) $c \in C^*(A, FM)$ there exists a unique (notation: $\exists!$) $n \in W^*(EA, M)$ such that there is a commutative diagram



$\forall c \in C^*(A, FM)$ $\exists! n \in W^*(EA, M)$

The analogous property for the back adjunction is obtained by interchanging the quantifiers "$\forall$" and "$\exists!$", interchanging C^* and W^*, and reversing some arrows:

$\forall n \in W^*(EA, M)$ $\exists! c \in C^*(A, FM)$



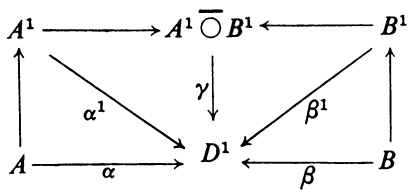
5.4. For any algebra A over C, define $A^1 = A$ if $1 \in A$ and $A^1 = C \times A$ otherwise. As usual, the notation $A \lhd A^1$ indicates that algebraically A is an ideal of A^1 (not necessarily closed). For nontopological algebras A, B their algebraic tensor product is defined as $A \otimes B = A^1 \bigcirc B + A \bigcirc B^1$.

5.5. For any C^*-homomorphism $\alpha: A \to B$, define another C^*-homomorphism $\alpha^1: A^1 \to B^1$ by $\alpha^1 = \alpha$ if $1 \in A$, and $\alpha^1(c, a) = ce + \alpha a$ if $1 \notin A$ and $A^1 = C \times A$, where

e is the identity of B^1. Note that the restriction of α^1 to A is α, but that the kernel of α^1 may be bigger than that of α.

If A and B are C^*-algebras, it should be recalled that $A \mathbin{\overline{\bigcirc}} B$ denotes the closure of $A \bigcirc B$ in the norm ρ (see 2.1) irrespective of whether A and, or, B do or do not contain identities. Since $A^1 \mathbin{\overline{\bigcirc}} B, A \bigcirc B^1 \lhd A^1 \mathbin{\overline{\bigcirc}} B^1$, and since a sum of closed ideals is closed in C^*-algebras, $A^1 \mathbin{\overline{\bigcirc}} B + A \mathbin{\overline{\bigcirc}} B^1$ is also a closed ideal in $A^1 \mathbin{\overline{\bigcirc}} B^1$. Define $A \mathbin{\overline{\otimes}} B = A \mathbin{\overline{\bigcirc}} B^1 + A^1 \mathbin{\overline{\bigcirc}} B$.

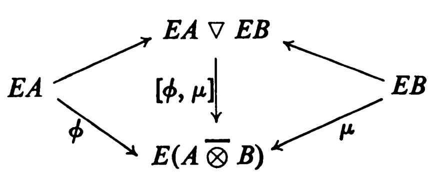
For any C^*-algebra A (with $1 \in A$ or $1 \notin A$), define $A_1 = \boldsymbol{C} \times A$. The categorical C^*-tensor product in the category of all C^*-algebras with or without identities, where homomorphisms need not preserve identities even when they exist, is $A \mathbin{\overline{\bigcirc}} B_1 + A_1 \mathbin{\overline{\bigcirc}} B$. Suppose $\alpha: A \to D$, $\beta: B \to D$ are C^*-maps with pointwise commuting images in D. Then α and β have natural extensions $\alpha_1: A_1 \to D_1$, $\beta_1: B_1 \to D_1$. There is an extension $\gamma: A_1 \mathbin{\overline{\bigcirc}} B_1 \to D_1$ and a commutative diagram



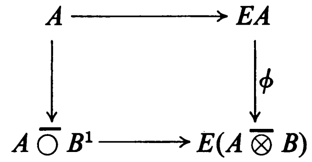
Since $\gamma(A_1 \mathbin{\overline{\bigcirc}} B) = \beta(B) + \gamma(A \mathbin{\overline{\bigcirc}} B)$, it follows that $\gamma(A \mathbin{\overline{\bigcirc}} B_1 + A_1 \mathbin{\overline{\bigcirc}} B) \subseteq D \subset D_1$. Now simply define $A \mathbin{\overline{\bigcirc}} B_1 + A_1 \mathbin{\overline{\bigcirc}} B \to D$ to be the restriction of γ to $A \mathbin{\overline{\otimes}} B$ and corestriction to D.

5.6. The nonfunctorial nature of the map $\alpha \to \alpha^1$ has to be clarified, in order to show why "$\_ \mathbin{\overline{\otimes}} \_$" is not a functor on the category $\boldsymbol{C^*}$. In general, if $\alpha: A \to B$ and $\beta: B \to D$, then $\beta^1\alpha^1 \neq (\beta\alpha)^1$. For example, take $A \neq A^1$, $B = B^1$, and $D = D^1$, but $\beta(1) \neq e$, where e is the identity of D. Then $(\beta\alpha)^1(1, 0) = e$, while $(\beta^1\alpha^1)(1, 0) = (\beta\alpha^1)(1, 0) = \beta(1) \neq e$. Consider the class of all C^*-homomorphisms $\alpha: A \to B$ with the property that if both algebras A and B have identities, then $\alpha(1) = 1$. If $1 \notin A$ or $1 \notin B$ or both, α is arbitrary. This class of maps is not closed under composition. However, if α, β, and $\beta\alpha$ belong to this class, then a straightforward verification of eight separate cases shows that $\beta^1\alpha^1 = (\beta\alpha)^1$.

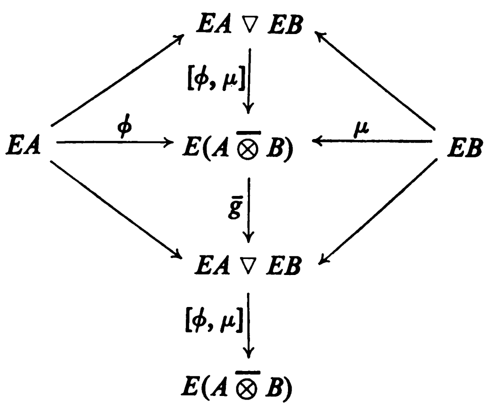
5.7. Theorem II. *For any C^*-algebras A, B (with or without identities) there exist monics ϕ, μ and a commutative diagram where $[\phi, \mu]$ is an isomorphism*



Proof. Since $A \overline{\bigcirc} B^1 \subset A \overline{\otimes} B \subset E(A \overline{\otimes} B)$ by the universal property of E, there exists a unique normal map ϕ and a commutative diagram



Similarly, let μ: $EB \to E(A \overline{\otimes} B)$. Since A, $B \subset (A \overline{\bigcirc} B^1 + A^1 \overline{\bigcirc} B)$ commute pointwise, so do also EA, $EB \subset E(A \overline{\bigcirc} B^1 + A^1 \overline{\bigcirc} B) \equiv E(A \overline{\otimes} B)$. Thus ϕ and μ define a unique map $[\phi, \mu]$: $EA \triangledown EB \to E(A \overline{\otimes} B)$. By the universal property of $-\overline{\otimes}-$, the functions $A \to EA \to EA \triangledown EB$ and $B \to EA \triangledown EB$ define a unique map $g: A \overline{\otimes} B \to EA \triangledown EB$. By the universal property of E, g factors through $\bar{g}$: $E(A \overline{\otimes} B) \to EA \triangledown EB$. Thus there is a commutative diagram



Both $\bar{g}[\phi, \mu]$ and the identity on $EA \triangledown EB$ are normal maps. By the uniqueness for $-\triangledown-$, it follows that $\bar{g}[\phi, \mu] = 1$. By the uniqueness of E, there is at most one normal map of $E(A \overline{\otimes} B) \to E(A \overline{\otimes} B)$ leaving $A \overline{\otimes} B$ pointwise fixed. Since $[\phi, \mu]\bar{g}$ is such a map, $[\phi, \mu]\bar{g} = 1$. Thus $[\phi, \mu]$ is an isomorphism whose inverse is $\bar{g}$.

The next corollary follows immediately from the fact that $E(A \overline{\otimes} B) \simeq EA \triangledown EB$.

Corollary 5.8. *For any C^*-algebras A and B, there is a σ-dense embedding $A \overline{\otimes} B \subset EA \triangledown EB$.*

5.9. Remarks. 1. The previous proposition implies the rather unexpected result that $E(A \overline{\bigcirc} B) \neq EA \triangledown EB$ in case one of A or B does not have an identity.

2. Presently, there is no direct proof of the previous corollary. It does not seem possible to show directly that $(A_* \triangledown B_*)^\perp \cap A \overline{\otimes} B = \{0\}$ in $(A \overline{\otimes} B)^{**}$, or, equivalently, that $A_* \triangle B_*$ separates the points of $A \overline{\otimes} B$.

3. The last proposition also shows that

$$(A \overline{\otimes} B)^{**} \cong (A^{**} \overline{\otimes} B^{**})^{**}/(A^* \triangle B^*)^\perp.$$

The conditions 5.10(i)(a), (b) (and (c)) below are the same as those in the definition of a multiplicative category (with an identity object), see [8, p. 3, Definition 2.1(a) and (b)].

5.10. If A, B and D are W^*-algebras, and $\boldsymbol{C}$ the complex numbers, then

(i)(a) $(A \bigtriangledown B) \bigtriangledown D \cong A \bigtriangledown (B \bigtriangledown D)$;

(b) $A \bigtriangledown B \cong B \bigtriangledown A$;

(c) $\boldsymbol{C} \bigtriangledown A \cong A$.

(ii) Consequently $(\boldsymbol{W^*}, \bigtriangledown)$ and $(\mathfrak{B}, \bigtriangleup)$ are multiplicative categories.

Proof. (i) Conclusions (a), (b) and (c) hold, first in the category of rings (with or without identities) with "$\_\bigtriangledown\_$" replaced by "$\_\otimes\_$" (see 5.4); and, secondly, for the category $\boldsymbol{C^*}$ of C^*-algebras if "$\_\overline{\otimes}\_$" is used in place of "$\_\bigtriangledown\_$". Since the above conclusions hold for either the algebraic or C^*-tensor products, and since either of the latter is σ-dense in the W^*-tensor product, (i) follows.

(ii) The restriction and corestriction of $\alpha \to \alpha^1$ to $\boldsymbol{C^*1}$ is just the identity functor. Thus $\_\overline{\otimes}\_$ is just equal to $\_\overline{\bigcirc}\_$. Thus, clearly, $\_\overline{\bigcirc}\_$ is a two variable functor $\boldsymbol{C^*1} \times \boldsymbol{C^*1} \to \boldsymbol{C^*1}$. Hence "$\_\bigtriangledown\_$" is a functor for $\boldsymbol{W^*}$ and $(\boldsymbol{W^*}, \bigtriangledown)$ is a multiplicative category. Since $\mathfrak{B}$ is dual to $\boldsymbol{W^*}$, also $(\mathfrak{B}, \bigtriangleup)$ is a multiplicative category.

The previous results are summarized in the next theorem. As a special case, the next theorem is applicable to the categories of C^*-algebras and W^*-algebras with identity and identity preserving morphisms.

5.11. THEOREM III. *Consider the categories* $\boldsymbol{C^*}$, $\boldsymbol{W^*}$, *and* $\mathfrak{B}$ *of* C^**-algebras* (*with or without identity*), *von Neumann algebras, and preduals. Let* $\boldsymbol{C^*1} \subset \boldsymbol{C^*}$ *be the full subcategory of* $\boldsymbol{C^*}$ *containing all* C^**-algebras with identity.* (*Note that the morphisms of* $\boldsymbol{C^*1}$ *need not preserve identities.*) *Let* E: $\boldsymbol{C^*} \to \boldsymbol{W^*}$ *be the functor that assigns to a* C^**-algebra* A *its universal von Neumann enveloping algebra* $EA = A^{**}$, *while* P: $\boldsymbol{W^*} \to \mathfrak{B}$ *is the contravariant functor where* PM *is the predual of a* W^**-algebra* M. *Consider the bifunctors* $\overline{\otimes}$, $\bigtriangledown$, *and* $\bigtriangleup$ (*as defined in* 5.6, 4.2, *and* 2.5) *on* $\boldsymbol{C^*1}$, $\boldsymbol{W^*}$, *and* $\mathfrak{B}$. *Then*

(i) $(\boldsymbol{C^*1}, \overline{\otimes})$, $(\boldsymbol{W^*}, \bigtriangledown)$, *and* $(\mathfrak{B}, \bigtriangleup)$ *are multiplicative categories.*

(ii) E: $(\boldsymbol{C^*1}, \overline{\otimes}) \to (\boldsymbol{W^*}, \bigtriangledown)$, *and* P: $(\boldsymbol{W^*}, \bigtriangledown) \to (\mathfrak{B}, \bigtriangledown)$ *are multiplicative functors of multiplicative categories* (*where* P *is a duality*).

For the next corollary, the reader should be aware that in any one of our three categories $(\boldsymbol{C^*1}, \overline{\otimes})$, $(\boldsymbol{W^*}, \bigtriangledown)$, or $(\mathfrak{B}, \bigtriangleup)$ the following concepts (see [8] and [13]) are applicable: (1) an algebra over a multiplicative category, (2) a morphism of algebras (in the sense of (1)), (3) cogebra, (4) morphism of cogebras, (5) bigebra, (6) morphism of bigebras, and (7) middle four interchange. Also, for a multiplicative category with identity, (1)–(6) have appropriate modifications. In $\boldsymbol{W^*}$, in addition, all morphisms have to be normal.

5.12. COROLLARY TO THEOREM III. *The functor* E: $(\boldsymbol{C^*1}, \overline{\otimes}) \to (\boldsymbol{W^*}, \bigtriangledown)$ *preserves algebras, cogebras, and bigebras. The functor* P: $(\boldsymbol{W^*}, \bigtriangledown) \to (\mathfrak{B}, \bigtriangleup)$ *transforms* *algebras → cogebras, cogebras → algebras, bigebras → bigebras.*

Parallel statements hold for the categories with identities and identity preserving morphisms.

6. **Examples and applications.** A drawback, but at the same time a challenging aspect of the present tensor product $\_\nabla\_$, is that it can very rarely be computed in a rigorous way in concrete examples, unless one is willing to make some assumptions.

First, the new tensor product $\_\nabla\_$ is compared with the usual one.

6.1. Suppose $A=A''\subset\mathscr{L}(H)$ and $B=B''\subset\mathscr{L}(K)$ are von Neumann algebras. Frequently, the double commutant $(A \bigcirc B)''\subset\mathscr{L}(H\overline{\otimes} K)$ of the algebraic tensor product is referred to as the von Neumann tensor product. Denote the operator norm closure of $A \bigcirc B$ by $A \square B$. Let $A_* \square B_*\subset A^* \square B^*\subset(A \square B)^*$ denote the closure of the algebraic tensor products in the dual norm of the operator norm on $\mathscr{L}(H\overline{\otimes} K)$. As before, $(A_* \square B_*)^{\perp}\subset(A \square B)^{**}$ is the annihilator of the translation invariant norm closed subspace $A_* \square B_*$. In [10, p. 3.17, Theorem 2.3] it is shown that $(A \bigcirc B)''\cong(A \square B)^{**}/(A_* \square B_*)^{\perp}$. By the universal property of $\_\nabla\_$, there is a normal map $\lambda: A \nabla B\to(A \bigcirc B)''$. The image $\lambda(A \nabla B)$ of a von Neumann algebra $A \nabla B$ under a normal homomorphism λ is a von Neumann subalgebra of $(A \bigcirc B)''$. But $A \bigcirc B\subset\lambda(A \bigcirc B)\subseteq(A \bigcirc B)''$, and $(A \bigcirc B)''$ is the smallest von Neumann subalgebra containing $A \bigcirc B$. Therefore $\lambda(A \nabla B)=(A \bigcirc B)''$.

Next, it is shown that $A \nabla B$ contains a copy of $A \square B$.

6.2. For any normal map λ of any two von Neumann algebras, $A \nabla B=K\oplus P$, where K is the kernel of λ, and where K and P are von Neumann ideals of $A \nabla B$. I.e. arbitrary least upper bounds of sets of positive elements in K, P agree with those in $A \nabla B$. Then $P\cong(A \bigcirc B)''$. Let $J=\lambda^{-1}(A \square B)$. Since $K\subset J$, $J=K\oplus(J\cap P)$. Also $A \bigcirc B\subset J$ and $J\cap P\cong A \square B$. Since λ is the identity on $A \bigcirc B$, $A \bigcirc B\cap K=\{0\}$ and the projection $\rho: A \bigcirc B\to J\cap P$ is one-to-one. A copy of $A \square B$ may be found inside $A \nabla B$ as follows:

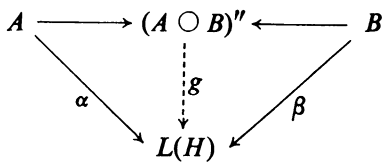
$$A \bigcirc B\cong \rho(A \bigcirc B)\subset J\cap P\cong A \square B.$$

Note that $A \square B$ does not contain the canonical copy of $A \bigcirc B$.

It will be shown that $\lambda(A \overline{\bigcirc} B)=A \square B$. Since $A \bigcirc B\subset J$, since J is norm closed, and since the closure of $A \bigcirc B$ in $A \nabla B$ is $A \overline{\bigcirc} B$, it follows that $A \overline{\bigcirc} B\subseteq J$. Thus $A \bigcirc B\subset\lambda(A \overline{\bigcirc} B)\subseteq A \square B$. However, $\lambda(A \overline{\bigcirc} B)$ is norm closed in $A \square B$, while $A \square B$ is the norm closure of $A \bigcirc B$. Thus $\lambda(A \overline{\bigcirc} B)=A \square B$.

The next example shows that the usual tensor product of von Neumann algebras and $\_\nabla\_$ are different.

6.3. EXAMPLE. Consider a type II factor $A\subset\mathscr{L}(H)$. The commutant $B=A'$ of A is also a type II factor. It suffices to show that $(A \bigcirc B)''$ does not have the universal property. It follows from [10, III, p. 3.40, Theorem 4.3] that also $(A \bigcirc B)''$ is of type II. Also, $(A \bigcirc B)''$ is a factor. If α and β are the natural inclusions, suppose that the following commutative diagram can be completed by a normal homomorphism g



The image of any W^*-algebra under a normal map is a von Neumann subalgebra. Then $g((A \bigcirc B)'')$ is a von Neumann subalgebra of $\mathscr{L}(H)$. However, $A, B \subset g((A \bigcirc B)'')$ and the von Neumann subalgebra generated by A and B is all of $\mathscr{L}(H)$. Thus $g((A \bigcirc B)'') = \mathscr{L}(H)$. However, $\mathscr{L}(H)$ is of type I, while $g((A \bigcirc B)'')$ is of type II, a contradiction. Thus there cannot exist such a map g.

Since $A \bigtriangledown B \not\cong (A \bigcirc B)''$, the map $\lambda: A \bigtriangledown B \to (A \bigcirc B)''$ of 6.1 must have a nontrivial kernel. If A and B are type II finite factors, then they are algebraically simple. Also $(A \bigcirc B)''$ is known to be algebraically simple. Thus this same example also serves to illustrate that even though A and B are algebraically simple, $A \bigtriangledown B$ need not be.

6.4. Any commutative von Neumann algebra M is of the form $M = L^\infty(X)$ on some measure space with $M_* = L^1(X)$. Consider a second commutative von Neumann algebra $N = L^\infty(Y)$. Denote the least and greatest cross norms, as usual, by λ and γ. The norm ρ used to form $M \mathbin{\overline{\otimes}} N$ is λ. Thus the norm on $M^* \mathbin{\overline{\otimes}} N^*$ is $\rho' = \lambda' = \gamma$ and $M_* \mathbin{\overline{\otimes}} N_* = L^1(X) \mathbin{\overline{\otimes}} L^1(Y) \cong L^1(X \times Y)$. In general, the inclusion $M \mathbin{\overline{\otimes}} N \subset L^\infty(X \times Y)$ is proper. Thus $(M \mathbin{\overline{\otimes}} N)^* \cong L^\infty(X \times Y)^*/(M \mathbin{\overline{\otimes}} N)^\perp$ where $L^\infty(X \times Y)^*$ is the vector space of regular Borel measures on the maximal ideal space of $L^\infty(X \times Y)$.

6.5. Assume $M_* \mathbin{\overline{\otimes}} N_* = M_* \bigtriangleup N_*$. Then $M \bigtriangledown N = (M_* \mathbin{\overline{\otimes}} N_*)^* = L^\infty(X \times Y)$. Thus $M \mathbin{\overline{\otimes}} N \neq M \bigtriangledown N$. Here $M \bigtriangledown N = (M \bigcirc N)''$ if $L^\infty(X) \subset \mathscr{L}(L^2(X))$ acts by multiplication. As Hilbert spaces, $L^2(X) \mathbin{\overline{\otimes}} L^2(Y) \cong L^2(X \times Y)$. It is easily seen that

$$(L^\infty(X) \bigcirc L^\infty(Y))' = (L^\infty(X) \bigcirc L^\infty(Y))'' = L^\infty(X \times Y).$$

6.6. If $X = Y = G$ is a locally compact group with the left invariant measure, then, as is well known, $L^\infty(G)$ is a cogebra in $(\boldsymbol{W^*}, \bigtriangledown)$ under the comultiplication $c: L^\infty(G) \to L^\infty(G \times G)$, where $(cf)(y, z) = f(yz)$ for $y, z \in G$. Then P transforms it into an ordinary Banach algebra $P(L^\infty(G)) = L^1(G)$ in $\mathfrak{B}$ where the multiplication is the usual convolution.

The objective of the next example is to verify directly some of the previous results by a direct computation.

Example 6.7. Consider a Hilbert space H, the algebra $\mathscr{L}$ of all bounded operators, and the three ideals—the trace-class, Schmidt-class, and compact operators $\mathscr{T} \subset \mathscr{S} \subset \mathscr{L}\mathscr{C} \subset \mathscr{L}$ respectively. If $\{e(i)\} \subset H$ is an arbitrary orthonormal basis then recall that $P \in \mathscr{S}$ if and only if $\sum \|Pe(i)\|^2 < \infty$ and that $\mathscr{T} = \mathscr{S}^2 = \{PQ \mid P, Q \in \mathscr{S}\}$.

For $T \in \mathscr{T}$, the trace $\operatorname{tr}(T)$ of T is $\operatorname{tr}(T) = \sum (Te(i), e(i))$. Then $\mathscr{L}\mathscr{C}^* = \mathscr{T}$ and $\mathscr{T}^* = \mathscr{L}$, where for $N \in \mathscr{L}$ and $T \in \mathscr{T}$, $\langle T, N\rangle = \operatorname{tr}(NT)$.

Suppose $D = \mathscr{L} = D_*^*$, where $D_* = \mathscr{T}$. Then $D^* = D_* \oplus eD^*$, where $D_*^{\perp} = D^{**}e \subset D^{**}$. Every element $f \in D^*$ is uniquely of the form $f = T + \delta$ with $\delta \in eD^*$ and $T \in \mathscr{T}$ where T is the restriction $T = f|\mathscr{L}\mathscr{C}$. Since $\delta|\mathscr{L}\mathscr{C} \in \mathscr{L}\mathscr{C}^* = \mathscr{T}$, and since $\mathscr{T} \cap eD^* = \{0\}$, it is reasonable to suppose and indeed can be shown [12, p. 50, Theorem 5] that $\langle \delta, \mathscr{L}\mathscr{C}\rangle = \{0\}$. Thus the singular part eD^* of D^* vanishes on $\mathscr{L}\mathscr{C}$.

Then $D^{**} = D_*^{\perp} \oplus D^{**}(1-e)$. By 3.3(iii) and (v), $D^{**}(1-e) \cong D_*^* = D$. Here the latter conclusion may be verified directly, since $D_* = \mathscr{T}$ and we know that $\mathscr{T}^* = \mathscr{L} = D$.

Let us interpret the conclusion in 3.6 that $\eta D \cap D^{**}e = \{0\}$. Suppose $B \in \mathscr{L}$ with $\eta B \in D^{**}e$. But by definition of e, $D^{**}e = D_*^{\perp} = \mathscr{T}^{\perp}$. Thus for all $T \in \mathscr{T}$,

$$\langle \eta B, T\rangle = \langle T, B\rangle = \operatorname{tr}(BT) = 0.$$

But it is shown in [12, p. 45, Lemma 1] that $\operatorname{tr}(BX) = 0$ for all operators $X \in \mathscr{T}$ of rank one if and only if $B = 0$. Thus we have verified directly that $\eta D \cap D^{**}e = \{0\}$.

The objective of the next example is to illustrate the conclusion $\eta A \cap A^{**}e = \{0\}$ in Proposition 3.5.

EXAMPLE 6.8. Consider $A = \mathscr{L}(H)$ for an infinite dimensional Hilbert space H. Every functional $f \in A_*$—the predual of A—is obtained from two square summable sequences of vectors $\{x(n)\}$, $\{y(n)\} \subseteq H$ by $f(T) = \sum (Tx(n), y(n))$ for $T \in A$ (see [10, III, p. 3.7, Proposition 1.1] or [20, p. 21, Lemma 6]). If $\eta A \subset A^{**}$, then $\eta(T)\colon A^* \to C$ and $\langle \eta T, f\rangle = f(T)$. If $0 \neq T \in A$ is arbitrary, then there is an f as above, in fact of the simple form $f(T) = (Tx, y)$, with $f(T) \neq 0$. Thus $\eta(T) \notin A_*^{\perp} = A^{**}e$ and $\eta A \cap A^{**}e = \{0\}$.

6.9. In conclusion, some unsolved problems are listed.

(1) Perhaps an intrinsic tensor product can be defined on all complex Banach spaces such that, when restricted to $\mathfrak{B}$, it agrees with $\_\triangle\_$.

(2) In the notation of 3.6, can the intersection $\eta A \cap A^{**}(1-e)$ be characterized?

(3) Under what conditions do the two tensor products agree:

$$(A \bigcirc B)'' \cong \frac{(A \square B)^{**}}{(A_* \square B_*)^{\perp}} = \frac{(A \overline{\otimes} B)^{**}}{(A_* \triangle B_*)^{\perp}} \cong A \triangledown B?$$

(4) To find examples with $A_* \overline{\otimes} B_* \neq A_* \triangle B_*$.

(5) Does $A_* \bigcirc B_*$ separate the points of $A \overline{\otimes} B$?

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